

AP[®] CALCULUS AB
2013 SCORING GUIDELINES

Question 6

Consider the differential equation $\frac{dy}{dx} = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

- (a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.
- (b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

(a) $\left. \frac{dy}{dx} \right|_{(x,y)=(1,0)} = e^0(3 \cdot 1^2 - 6 \cdot 1) = -3$

An equation for the tangent line is $y = -3(x - 1)$.

$$f(1.2) \approx -3(1.2 - 1) = -0.6$$

$$3 : \begin{cases} 1 : \frac{dy}{dx} \text{ at the point } (x, y) = (1, 0) \\ 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$$

(b) $\frac{dy}{e^y} = (3x^2 - 6x) dx$

$$\int \frac{dy}{e^y} = \int (3x^2 - 6x) dx$$

$$-e^{-y} = x^3 - 3x^2 + C$$

$$-e^{-0} = 1^3 - 3 \cdot 1^2 + C \Rightarrow C = 1$$

$$-e^{-y} = x^3 - 3x^2 + 1$$

$$e^{-y} = -x^3 + 3x^2 - 1$$

$$-y = \ln(-x^3 + 3x^2 - 1)$$

$$y = -\ln(-x^3 + 3x^2 - 1)$$

$$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Note: This solution is valid on an interval containing $x = 1$ for which $-x^3 + 3x^2 - 1 > 0$.

AP[®] CALCULUS AB
2012 SCORING GUIDELINES

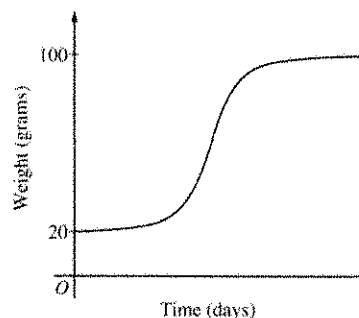
Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

- (a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.
- (b) Find $\frac{d^2B}{dt^2}$ in terms of B . Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.
- (c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



(a) $\left. \frac{dB}{dt} \right|_{B=40} = \frac{1}{5}(60) = 12$

$$\left. \frac{dB}{dt} \right|_{B=70} = \frac{1}{5}(30) = 6$$

Because $\left. \frac{dB}{dt} \right|_{B=40} > \left. \frac{dB}{dt} \right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5} \frac{dB}{dt} = -\frac{1}{5} \cdot \frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \leq B < 100$. A portion of the given graph is concave up.

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$$\int \frac{1}{100 - B} dB = \int \frac{1}{5} dt$$

$$-\ln|100 - B| = \frac{1}{5}t + C$$

Because $20 \leq B < 100$, $|100 - B| = 100 - B$.

$$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$$

$$100 - B = 80e^{-t/5}$$

$$B(t) = 100 - 80e^{-t/5}, \quad t \geq 0$$

2 : $\begin{cases} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{cases}$

2 : $\begin{cases} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{cases}$

5 : $\begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

AP[®] CALCULUS AB
2011 SCORING GUIDELINES

Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

- (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).
- (b) Find $\frac{d^2W}{dt^2}$ in terms of W . Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.
- (c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

(a) $\left. \frac{dW}{dt} \right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411 \text{ tons}$$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25} \frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$$\int \frac{1}{W - 300} dW = \int \frac{1}{25} dt$$

$$\ln|W - 300| = \frac{1}{25}t + C$$

$$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$$

$$W - 300 = 1100e^{\frac{1}{25}t}$$

$$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$$

$$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$$

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

AP[®] CALCULUS AB
2010 SCORING GUIDELINES

Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

- Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.
- Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.
- Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

(a) $f'(1) = \left. \frac{dy}{dx} \right|_{(1, 2)} = 8$

An equation of the tangent line is $y = 2 + 8(x - 1)$.

2: $\begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$

(b) $f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval $1 \leq x < 1.1$,

$$\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2) > 0 \text{ on this interval.}$$

Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

2: $\begin{cases} 1 : \text{approximation} \\ 1 : \text{conclusion with explanation} \end{cases}$

(c) $\frac{dy}{dx} = xy^3$

$$\int \frac{1}{y^3} dy = \int x dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + C$$

$$-\frac{1}{2 \cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$$

$$y^2 = \frac{1}{\frac{5}{4} - x^2}$$

$$f(x) = \frac{2}{\sqrt{5 - 4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

5: $\begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

AP[®] CALCULUS AB
2010 SCORING GUIDELINES (Form B)

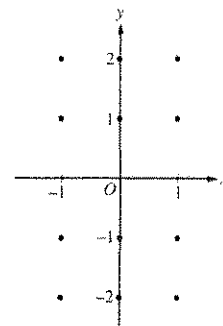
Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

(Note: Use the axes provided in the exam booklet.)

- (b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy -plane for which $y \neq 0$. Describe all points in the xy -plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.
- (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.



(a)

3 : { 1 : zero slopes
1 : nonzero slopes
1 : solution curve through $(0, -1)$

(b) $-1 = \frac{x+1}{y} \Rightarrow y = -x-1$

$\frac{dy}{dx} = -1$ for all (x, y) with $y = -x-1$ and $y \neq 0$

(c) $\int y \, dy = \int (x+1) \, dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0, -2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

1 : description

5 : { 1 : separates variables
1 : antiderivatives
1 : constant of integration
1 : uses initial condition
1 : solves for y

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

AP[®] CALCULUS AB
2008 SCORING GUIDELINES

Question 5

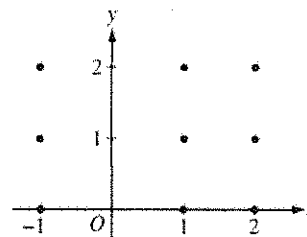
Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

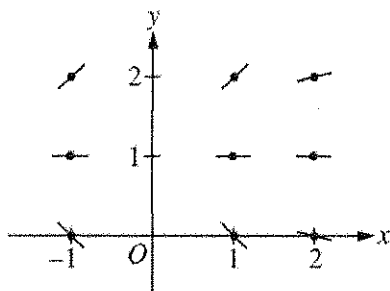
(Note: Use the axes provided in the exam booklet.)

- (b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

- (c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \rightarrow \infty} f(x)$.



(a)



2 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) $\frac{1}{y-1} dy = \frac{1}{x^2} dx$

$$\ln|y-1| = -\frac{1}{x} + C$$

$$|y-1| = e^{-\frac{1}{x}+C}$$

$$|y-1| = e^C e^{-\frac{1}{x}}$$

$$y-1 = k e^{-\frac{1}{x}}, \text{ where } k = \pm e^C$$

$$-1 = k e^{-\frac{1}{2}}$$

$$k = -e^{\frac{1}{2}}$$

$$f(x) = 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)}, x > 0$$

6 : $\begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(c) $\lim_{x \rightarrow \infty} 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

AP[®] CALCULUS AB
2008 SCORING GUIDELINES (Form B)

Question 6

Consider the closed curve in the xy -plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

- (a) Show that $\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}$.
- (b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.
- (c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.
- (d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x -axis? Explain your reasoning.

(a) $2x + 2 + 4y^3 \frac{dy}{dx} + 4 \frac{dy}{dx} = 0$

$$(4y^3 + 4) \frac{dy}{dx} = -2x - 2$$

$$\frac{dy}{dx} = \frac{-2(x+1)}{4(y^3+1)} = \frac{-(x+1)}{2(y^3+1)}$$

$$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{verification} \end{cases}$$

(b) $\left. \frac{dy}{dx} \right|_{(-2, 1)} = \frac{-(-2+1)}{2(1+1)} = \frac{1}{4}$

Tangent line: $y = 1 + \frac{1}{4}(x + 2)$

$$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$$

- (c) Vertical tangent lines occur at points on the curve where $y^3 + 1 = 0$ (or $y = -1$) and $x \neq -1$.

On the curve, $y = -1$ implies that $x^2 + 2x + 1 - 4 = 5$,
so $x = -4$ or $x = 2$.

Vertical tangent lines occur at the points $(-4, -1)$ and $(2, -1)$.

$$3 : \begin{cases} 1 : y = -1 \\ 1 : \text{substitutes } y = -1 \text{ into the} \\ \quad \text{equation of the curve} \\ 1 : \text{answer} \end{cases}$$

- (d) Horizontal tangents occur at points on the curve where $x = -1$ and $y \neq -1$.

The curve crosses the x -axis where $y = 0$.

$$(-1)^2 + 2(-1) + 0^4 + 4 \cdot 0 \neq 5$$

No, the curve cannot have a horizontal tangent where it crosses the x -axis.

$$2 : \begin{cases} 1 : \text{works with } x = -1 \text{ or } y = 0 \\ 1 : \text{answer with reason} \end{cases}$$

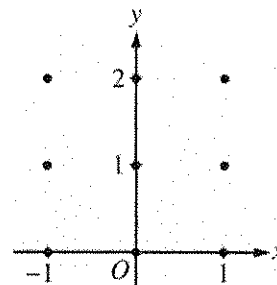
AP[®] CALCULUS AB
2007 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

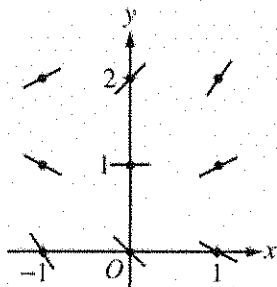
- (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

(Note: Use the axes provided in the exam booklet.)



- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Describe the region in the xy -plane in which all solution curves to the differential equation are concave up.
- (c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.
- (d) Find the values of the constants m and b , for which $y = mx + b$ is a solution to the differential equation.

(a)



2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

3 : $\begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

(c) $\left. \frac{dy}{dx} \right|_{(0,1)} = 0 + 1 - 1 = 0$ and $\left. \frac{d^2y}{dx^2} \right|_{(0,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

- (d) Substituting $y = mx + b$ into the differential equation:

$$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$$

Then $0 = m + \frac{1}{2}$ and $m = b - 1$; $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

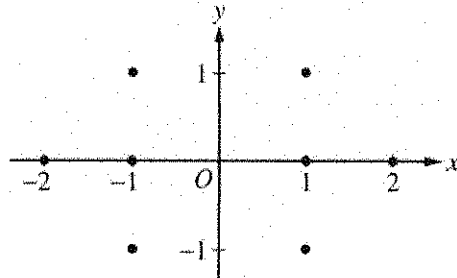
2 : $\begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$

**AP[®] CALCULUS AB
2006 SCORING GUIDELINES**

Question 5

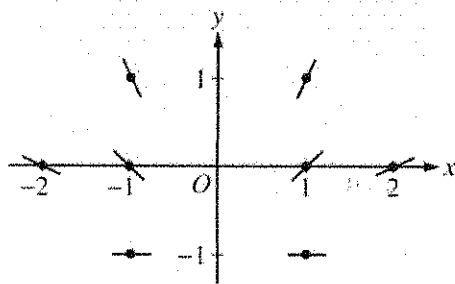
Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.
(Note: Use the axes provided in the pink exam booklet.)



- (b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

(a)



2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(b) $\frac{1}{1+y} dy = \frac{1}{x} dx$

$$\ln|1+y| = \ln|x| + K$$

$$|1+y| = e^{\ln|x| + K}$$

$$1+y = C|x|$$

$$2 = C$$

$$1+y = 2|x|$$

$$y = 2|x| - 1 \text{ and } x < 0$$

or

$$y = -2x - 1 \text{ and } x < 0$$

6 : $\left\{ \begin{array}{l} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{array} \right.$

7 : $\left\{ \begin{array}{l} \text{Note: max } 3/6 \text{ [1-2-0-0-0] if no constant of integration} \\ \text{Note: } 0/6 \text{ if no separation of variables} \end{array} \right.$

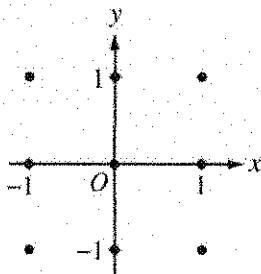
1 : domain

AP[®] CALCULUS AB
2006 SCORING GUIDELINES (Form B)

Question 5

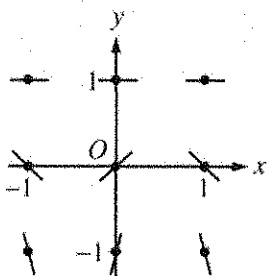
Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



- (b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c .
(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



2 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

- (b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

1 : $c = 1$

(c) $\frac{1}{(y-1)^2} dy = \cos(\pi x) dx$

$$-(y-1)^{-1} = \frac{1}{\pi} \sin(\pi x) + C$$

$$\frac{1}{1-y} = \frac{1}{\pi} \sin(\pi x) + C$$

$$1 = \frac{1}{\pi} \sin(\pi) + C = C$$

$$\frac{1}{1-y} = \frac{1}{\pi} \sin(\pi x) + 1$$

$$\frac{\pi}{1-y} = \sin(\pi x) + \pi$$

$$y = 1 - \frac{\pi}{\sin(\pi x) + \pi} \text{ for } -\infty < x < \infty$$

6 : $\begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables